

A Note on a Theorem of Parry

Chris Preston

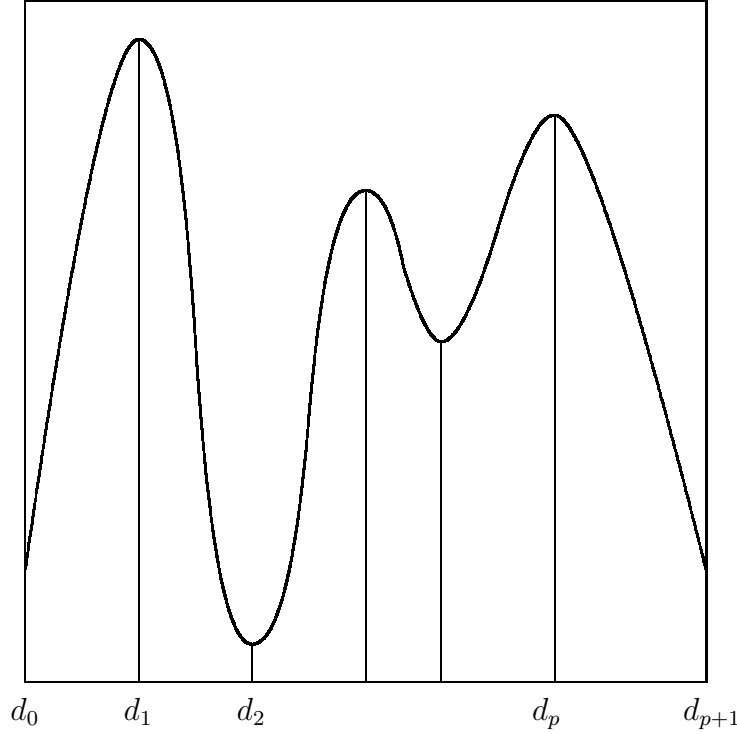
In [9] Parry shows that a topologically transitive continuous piecewise monotone mapping f with positive topological entropy $h(f)$ is conjugate to a uniformly piecewise linear mapping with slope $\exp(h(f))$. In this note we generalise Parry's result somewhat to what we call the class of essentially transitive mappings. This generalisation is of some interest in as much as for mappings with one turning point the converse also holds, i.e., a uniformly piecewise linear mapping g with one turning point and with slope $\beta > 1$ is essentially transitive. (In fact, g is topologically transitive if and only if the slope β lies in the interval $(\sqrt{2}, 2]$; if $\beta \in (1, \sqrt{2}]$ then g is only essentially transitive.) The proof of our generalisation relies heavily on a result from the kneading theory of Milnor and Thurston [5], [6], which states that a continuous piecewise monotone mapping f with positive topological entropy $h(f)$ is semi-conjugate to a uniformly piecewise linear mapping with slope $\exp(h(f))$. We show that if f is essentially transitive then this forces the semi-conjugacy to be a conjugacy.

Let $a, b \in \mathbb{R}$ with $a < b$ and put $I = [a, b]$. The set of continuous mappings which map the interval I back into itself will be denoted by $C(I)$. If $f \in C(I)$ and $n \geq 0$ then f^n will denote the n th iterate of f , i.e., $f^n \in C(I)$ is defined inductively by $f^0(x) = x$, $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for each $x \in I$, or, without arguments, by $f^0 = \text{id}_I$, $f^1 = f$ and $f^n = f \circ f^{n-1}$ for each $n \geq 2$.

Mappings $g, h \in C(I)$ are said to be *conjugate* if there exists a homeomorphism $\psi : I \rightarrow I$ (which in the present situation just means a continuous and strictly monotone mapping of I onto itself) such that $\psi \circ g = h \circ \psi$. Let $f \in C(I)$; a subset B of I is said to be *f-invariant* if $f(B) \subset B$, and the mapping f is said to be (*topologically*) *transitive* if whenever F is a closed f -invariant subset of I then either $F = I$ or the interior $\text{int}(F)$ of F is empty. There are several other standard conditions which are equivalent to that of being transitive; see, for example, Walters [13].

We are here interested in a special class of mappings from $C(I)$, namely the piecewise monotone mappings. A mapping $f \in C(I)$ is *piecewise monotone* if there exists $p \geq 0$ and $a = d_0 < d_1 < \dots < d_p < d_{p+1} = b$ such that f is strictly monotone on each of the intervals $[d_k, d_{k+1}]$, $k = 0, \dots, p$. Let f be piecewise monotone and suppose the minimal choice for the d_k 's has been made, i.e., so

that f is not monotone (or, equivalently, is not injective) on any open interval containing d_k for each $1 \leq k \leq p$; then d_1, \dots, d_p are called the *turning points* of f and the intervals $[d_k, d_{k+1}]$, $k = 0, \dots, p$ the *laps* of f .



The set of piecewise monotone mappings in $C(I)$ will be denoted by $M(I)$ and for each $f \in M(I)$ the set of turning points of f by $T(f)$. The mappings in $M(I)$ are closed under composition: If $f, g \in M(I)$ then $g \circ f \in M(I)$ and it is easy to see that $T(g \circ f) = \{x \in (a, b) : x \in T(f) \text{ or } f(x) \in T(g)\}$. In particular, $f^n \in M(I)$ for all $f \in M(I)$, $n \geq 1$.

Let $\ell(f)$ denote the number of laps of $f \in M(I)$, so $\ell(f) = \#(T(f)) + 1$; also let $h(f) = \inf_{n \geq 1} n^{-1} \log \ell(f^n)$, and thus $h(f) \geq 0$.

Lemma 1 $h(f) = \lim_{n \rightarrow \infty} n^{-1} \log \ell(f^n)$.

Proof If $f, g \in M(I)$ then $\ell(f \circ g) \leq \ell(f)\ell(g)$, since each lap of g contains at most $\ell(f)$ laps of $f \circ g$, and so in particular $\ell(f^{m+n}) \leq \ell(f^m)\ell(f^n)$ for all $m, n \geq 1$. Put $a_n = \log \ell(f^n)$; then $a_{m+n} \leq a_m + a_n$ for all $m, n \geq 1$, and hence, as is well-known $\lim_{n \rightarrow \infty} a_n/n = \inf_{n \rightarrow \infty} a_n/n$. \square

Misiurewicz and Szlenk [8] show that $h(f)$ is the topological entropy of f (which is why we denote this quantity by $h(f)$).

Let $\beta > 0$; a mapping $g \in M(I)$ is said to be *uniformly piecewise linear* with *slope* β if on each of its laps g is linear with slope β or $-\beta$. The result of Parry referred to in the title of this note (Theorem 1 in [9]) states that if $f \in M(I)$ is transitive then $h(f) > 0$ and f is conjugate to a uniformly piecewise linear mapping with slope $\beta = \exp(h(f))$. Parry's result will be generalised somewhat below to what we call the class of essentially transitive mappings in $M(I)$.

Let $f \in C(I)$; a closed set $C \subset I$ is called an *f-cycle* with *period* $m \geq 1$ if C is the disjoint union of non-trivial closed intervals B_0, \dots, B_{m-1} such that $f(B_{k-1}) \subset B_k$ for $k = 1, \dots, m-1$ and $f(B_{m-1}) \subset B_0$ (and so in particular C is f -invariant). An f -cycle C is said to be (*topologically*) *transitive* if whenever F is a closed f -invariant subset of C then either $F = C$ or $\text{int}(F) = \emptyset$. Thus f being transitive just means that the whole interval I is a transitive f -cycle (with period 1). For each subset $B \subset I$ put

$$E(B, f) = \{x \in I : f^n(x) \in B \text{ for some } n \geq 0\},$$

i.e., $E(B, f)$ consists of those points x for which some iterate $f^n(x)$ lies in the set B . The complement $I \setminus E(B, f)$ of this set is always f -invariant and if B is f -invariant then so is $E(B, f)$.

We call a mapping $f \in C(I)$ *essentially transitive* if there exists a transitive f -cycle C such that $I \setminus E(C, f)$ is countable.

Theorem 1 *If the mapping $f \in M(I)$ is essentially transitive then $h(f) > 0$ and f is conjugate to a uniformly piecewise linear mapping with slope $\exp(h(f))$.*

Proof This will follow directly from Theorem 2, Proposition 2 and Lemma 2. Theorem 2 is a result from the kneading theory of Milnor and Thurston [5], [6] and Lemma 2 is essentially already a part of Parry's result. Thus the only part which is new here is Proposition 2. \square

In particular, a transitive mapping is essentially transitive, and so Parry's result is a special case of Theorem 1. For mappings with one turning point the converse of Theorem 1 holds: In Proposition 3 we show that in this case each uniformly piecewise linear mapping with slope $\beta > 1$ is essentially transitive.

We should point out that an essentially transitive mapping involves a very special situation, as the follows result indicates:

Proposition 1 *Let $f \in M(I)$ and let C be an f -cycle such that $I \setminus E(C, f)$ is countable. Then the period of C is of the form 2^p for some $p \geq 0$. Moreover, there exists $q \geq 0$ such that each periodic point in $I \setminus E(C, f)$ has a period which divides 2^q , and each point in $I \setminus E(C, f)$ is eventually periodic.*

Proof This is Proposition 9.2 in Preston [10], the proof of which is based on an idea occurring in Block [1], [2], Misiurewicz [7], and in the proof of Šarkovskii's theorem (Šarkovskii [11], Štefan [12]) given in Block, Guckenheimer, Misiurewicz and Young [3]. \square

Let $V(I) = \{\psi \in C(I) : \psi \text{ is increasing and surjective}\}$ (where increasing means only that $\psi(x) \geq \psi(y)$ whenever $x \geq y$). A pair (ψ, g) with $\psi \in V(I)$ and $g \in M(I)$ is called a *reduction* (or *semi-conjugacy*) of $f \in M(I)$ if $\psi \circ f = g \circ \psi$.

Theorem 2 *Let $f \in M(I)$ with $h(f) > 0$. Then there exists a reduction (ψ, g) of f such that g is uniformly piecewise linear with slope $\exp(h(f))$.*

Proof This can be found in Milnor and Thurston [5], [6]. A modification of their proof (not using any complex analysis) is given at the end of this note. \square

Lemma 2 *Let $f \in M(I)$; if there exists a transitive f -cycle then $h(f) > 0$. In particular, $h(f) > 0$ whenever f is essentially transitive.*

Proof Let C be a transitive f -cycle; put $m = \text{per}(C)$, let B be one of the m components of C , and let g be the restriction of f^m to B , which means that $g \in M(B)$. Then $\ell(g^n) \leq \ell(f^{mn})$ for each $n \geq 0$ and thus by Lemma 1

$$h(f) = \lim_{n \rightarrow \infty} \frac{1}{mn} \log \ell(f^{mn}) \geq \lim_{n \rightarrow \infty} \frac{1}{mn} \log \ell(g^n) = \frac{1}{m} h(g) .$$

But $g \in M(B)$ is transitive and so it is enough to show that $h(f) > 0$ for each transitive $f \in M(I)$ (which is already part of Parry's result). This holds because a transitive mapping $f \in M(I)$ contains some kind of 'horse-shoe': There exist $p \geq 1$ and $D, E \subset I$ with $D \cap E = \emptyset$ such that $f^p(D) \cap f^p(E) \supset D \cup E$. Thus f^p is at least 2 to 1 (i.e., $\#((f^p)^{-1}(x)) \geq 2$ for each $x \in I$), and so f^{pn} is at least 2^n to 1 for each $n \geq 1$. It follows that $\ell(f^{pn}) \geq 2^n$ for all $n \geq 1$, and hence that

$$h(f) = \lim_{n \rightarrow \infty} \frac{1}{pn} \log \ell(f^{pn}) \geq \frac{1}{p} \log 2 > 0 .$$

The existence of such a 'horse-shoe' follows, for example, from Theorem 2.1 in Preston [10], which states that a transitive mapping in $M(I)$ is either exact or semi-exact. However, the reader can probably establish the existence directly without too much trouble. \square

Proposition 2 *If (ψ, g) is a reduction of an essentially transitive $f \in M(I)$ then ψ must be a homeomorphism, and so in particular f and g are conjugate.*

Proof For each $\psi \in V(I)$ put

$$\text{supp}(\psi) = \{x \in I : \psi(J) \text{ is a non-trivial interval for each open interval } J \subset I \text{ containing } x\},$$

which means of course that

$$I \setminus \text{supp}(\psi) = \{x \in I : \text{there exists an open interval } J \subset I \text{ containing } x \text{ such that } \psi(J) \text{ consists of the single point } \psi(x)\},$$

and note that ψ is a homeomorphism if and only if $\text{supp}(\psi) = I$.

Lemma 3 *For each $\psi \in V(I)$ the set $\text{supp}(\psi)$ is non-empty and perfect (i.e., it is closed and contains no isolated points).*

Proof Clearly $\text{supp}(\psi)$ is closed. Moreover, ψ is constant on each connected component of $I \setminus \text{supp}(\psi)$. (Let J be such a component and consider $c, d \in J$ with $c < d$. For each $x \in [c, d]$ there exists an open interval J_x containing x on which ψ is constant. By compactness $[c, d]$ is covered by finitely many of these intervals and hence $\psi(c) = \psi(d)$.) In particular, this means that $\text{supp}(\psi)$ is non-empty and perfect (since an isolated point of $\text{supp}(\psi)$ would be the common end-point of two connected components of $I \setminus \text{supp}(\psi)$). \square

In fact, each non-empty perfect subset of I is of the form $\text{supp}(\psi)$, i.e., if D is a non-empty perfect subset of I then there exists a $\psi \in V(I)$ with $\text{supp}(\psi) = D$. (This is a classical result in real analysis, and can be found, for example, in Carathéodory [4]. A proof is also given in Preston [10], Proposition 11.1.)

For each $f \in M(I)$ put $S(f) = T(f) \cup \{a, b\}$. A subset $A \subset I$ will be called *f-almost-invariant* if $f(A \setminus S(f)) \subset A$.

Lemma 4 *Let (ψ, g) be a reduction of a mapping $f \in M(I)$. Then $\text{supp}(\psi)$ is f -invariant and $I \setminus \text{supp}(\psi)$ is f -almost-invariant.*

Proof Let $x \in I$ with $f(x) \notin \text{supp}(\psi)$; there thus exists an open interval J containing $f(x)$ such that $\psi(J)$ consists of the single point $y = \psi(f(x))$. Hence $f^{-1}(J)$ is a neighbourhood of x and so there exists an open interval K containing x with $K \subset f^{-1}(J)$. Then $g(\psi(K)) = \psi(f(K)) \subset \psi(J) = \{y\}$, since $f(K) \subset J$ and $g \circ \psi = \psi \circ f$. But $\psi(K)$ is connected and $g^{-1}(\{y\})$ is finite, and therefore $\psi(K)$ must consist of the single point $\{\psi(x)\}$, i.e., $x \notin \text{supp}(\psi)$. This shows that $\text{supp}(\psi)$ is f -invariant.

Now let $x \in (I \setminus \text{supp}(\psi)) \setminus S(f)$; there thus exists an open interval J containing x with $J \cap S(f) = \emptyset$ such that $\psi(J)$ consists of the single point $y = \psi(x)$.

Therefore $f(J)$ is an open interval containing $f(x)$ (since $J \cap S(f) = \emptyset$). But $\psi(f(J)) = g(\psi(J)) = g(\{y\})$, since $g \circ \psi = \psi \circ f$, and so $\psi(f(J))$ consists of the single point $\psi(\{x\})$, i.e., $f(x) \in I \setminus \text{supp}(\psi)$. This shows that $I \setminus \text{supp}(\psi)$ is f -almost-invariant. \square

For each $f \in M(I)$ let $\mathcal{D}(f)$ denote the set of those non-empty perfect subsets of I which are both f -invariant and have an f -almost-invariant complement. If (ψ, g) is a reduction of f then by Lemmas 3 and 4 $\text{supp}(\psi) \in \mathcal{D}(f)$. In fact, the converse also holds: For each $\psi \in V(I)$ with $\text{supp}(\psi) \in \mathcal{D}(f)$ there exists a unique $g \in M(I)$ such that (ψ, g) is a reduction of f . (This is part of Theorem 5.1 in Preston [10].)

If C is an f -cycle then let C° denote the set obtained by removing the two end-points from each component of C , so if m is the period of C then $\partial C = C \setminus C^\circ$ consists of exactly $2m$ points. The set C° is not necessarily f -invariant but it is easy to see that it is f -almost-invariant, which in turn easily implies that the open set $E(C^\circ, f)$ is f -almost-invariant. Of course, $E(C^\circ, f) \subset E(C, f)$, since $C^\circ \subset C$; moreover, $E(C, f) \setminus E(C^\circ, f)$ is countable, since it is a subset of the countable set $\{x \in I : f^n(x) \in \partial C \text{ for some } n \geq 0\}$. Thus $I \setminus E(C, f)$ is countable if and only if $I \setminus E(C^\circ, f)$ is.

Lemma 5 *Let $f \in M(I)$, $D \in \mathcal{D}(f)$ and C be a transitive f -cycle. Then either $E(C^\circ, f) \subset D$ or $E(C^\circ, f) \cap D = \emptyset$.*

Proof Let $U \subset I$ be open and f -almost-invariant. If J is a (maximal connected) component of U then $f(J \setminus S(f)) \subset U$ and it is easily checked that $f(J \setminus S(f))$ is connected (and in fact an open interval). There thus exists a unique component K of U such that $f(\overline{J}) \subset \overline{K}$. Iterating this then gives us that for each $n \geq 1$ there exists a unique component K of U such that $f^n(\overline{J}) \subset \overline{K}$. A component J of U is called *periodic* if $f^m(\overline{J}) \subset \overline{J}$ for some $m \geq 1$; the smallest such $m \geq 1$ is called the *period* of J . A component K of U is called *eventually periodic* if $f^n(\overline{K}) \subset \overline{J}$ for some periodic component J of U and some $n \geq 0$.

Now suppose $E(C^\circ, f) \not\subset D$; then $E(C^\circ, f) \cap (I \setminus D) \neq \emptyset$ and it easily follows that $U = C^\circ \cap (I \setminus D) \neq \emptyset$, so U is a non-empty f -almost-invariant open subset of I . Let J be a component of U which is not eventually periodic and for each $n \geq 0$ let J_n be the component of U with $f^n(\overline{J}) \subset \overline{J_n}$. Then the intervals $\{J_n\}_{n \geq 0}$ are disjoint and $F = \overline{\bigcup_{n \geq 1} J_n}$ is an f -invariant closed subset of C (since $f(\overline{J_n}) \subset \overline{J_{n+1}}$ for each $n \geq 1$). But $\text{int}(F) \neq \emptyset$ (since $J_1 \subset F$) and $F \neq C$ (since $J \subset C \setminus F$) and this contradicts the fact that C is transitive. Therefore each component of U is eventually periodic and in particular U contains a periodic component. Thus let J be a periodic component of U with period m ; then $K = \bigcup_{k=0}^{m-1} \overline{f^k(J)}$ is a closed f -invariant subset of C with $\text{int}(K) \neq \emptyset$, and hence $K = C$. But $C^\circ \cap D \subset K \setminus U$

and $K \setminus U$ is finite. Therefore $C^o \cap D = \emptyset$, because D is perfect. This implies that $E(C^o, f) \cap D = \emptyset$, since D is f -invariant. \square

We can now complete the proof of Proposition 2. Let C be transitive f -cycle such that $I \setminus E(C^o, f)$ is countable. By Lemmas 3 and 4 $\text{supp}(\psi) \in \mathcal{D}(f)$ and so by Lemma 5 either $E(C^o, f) \cap \text{supp}(\psi)$ is empty or $E(C^o, f) \subset \text{supp}(\psi)$. But if $E(C^o, f) \cap \text{supp}(\psi) = \emptyset$ then $\text{supp}(\psi)$ is countable, which is not possible since by the Baire category theorem a non-empty countable closed subset of I must contain an isolated point. Hence $E(C^o, f) \subset \text{supp}(\psi)$ and therefore $\text{supp}(\psi) = I$, since $E(C^o, f)$ is dense in I . This implies that ψ is a homeomorphism. \square

Proof of Theorem 1: Let $f \in M(I)$ be essentially transitive. Then by Lemma 2 $h(f) > 0$ and hence by Theorem 1 there exists a reduction (ψ, g) of f such that g is uniformly piecewise linear with slope $\exp(h(f))$. But by Proposition 2 ψ is a homeomorphism and therefore f and g are conjugate. \square

We next note a result of Misiurewicz and Szlenk [8] which provides us with an alternative method of calculating $h(f)$ for a mapping $f \in M(I)$. This will show in particular that if $g \in M(I)$ is uniformly piecewise linear with slope $\beta > 1$, then $h(g) = \log \beta$. For $f \in C(I)$ let

$$\text{Var}(f) = \sup \left\{ \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| : a = a_0 < x_1 < \cdots < x_n = b \right\}.$$

If $f \in M(I)$ and $a = d_0 < d_1 < \cdots < d_N < d_{N+1} = b$, where d_1, \dots, d_N are the turning points of f then clearly

$$\text{Var}(f) = \sum_{k=0}^N |f(d_{k+1}) - f(d_k)|.$$

In particular, if $g \in M(I)$ is uniformly piecewise linear with slope $\beta > 0$ then $\text{Var}(g) = (b - a)\beta$.

Theorem 3 *Let $f \in M(I)$; then $h(f) > 0$ holds if and only if*

$$\limsup_{n \rightarrow \infty} n^{-1} \log \text{Var}(f^n) > 0.$$

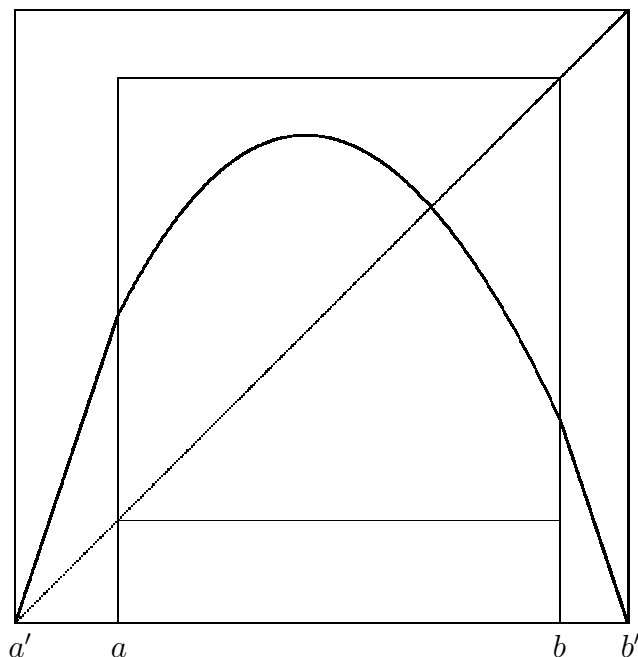
Moreover, if $h(f) > 0$ then

$$h(f) = \lim_{n \rightarrow \infty} n^{-1} \log \text{Var}(f^n).$$

Proof This is given in Misiurewicz and Szlenk [8]. \square

Let g be uniformly piecewise linear with slope $\beta > 0$. Then $\text{Var}(g^n) = (b-a)\beta^n$ for each $n \geq 1$, since g^n is uniformly piecewise linear with slope β^n , and thus $\lim_{n \rightarrow \infty} n^{-1} \log \text{Var}(g^n) = \log \beta$. Hence by Theorem 3 $h(g) = \log \beta$, provided $\beta > 1$.

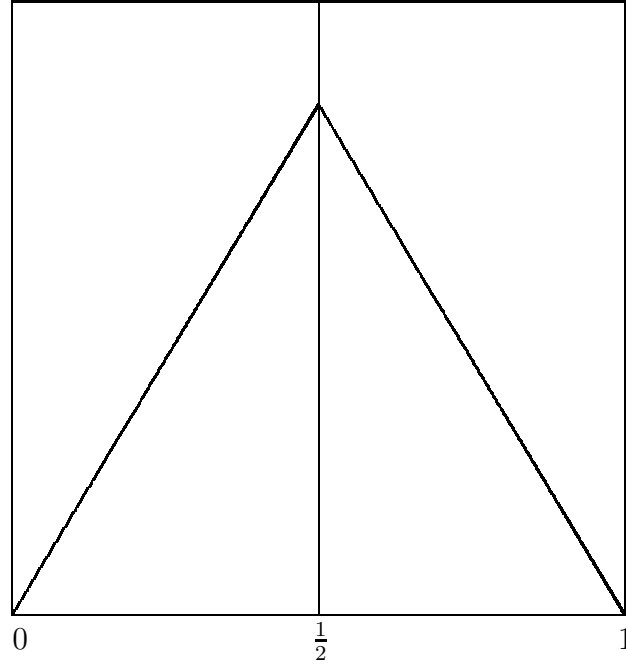
We now consider the case of a mapping $f \in M(I)$ having a single turning point, and without loss of generality it can be assumed that f takes on its maximum there. Moreover, it will be convenient to assume also that $f(\{a, b\}) \subset \{a, b\}$, i.e., that $f(a) = f(b) = a$. Note that it is always possible to reduce things to this case by extending the domain of definition of f to a larger interval. Moreover, this can be done in such a way that for each $x \in (a', a) \cup (b, b')$ the iterates of x end up in $[a, b]$ after finitely many steps.



Finally, again without loss of generality, assume that $I = [0, 1]$; let S denote the set of mappings $f \in M([0, 1])$ having exactly one turning point and for which $f(0) = f(1) = 0$.

For each $\beta \in (0, 2]$ there is exactly one mapping in S which is uniformly piecewise linear with slope β . This is the mapping u_β defined by

$$u_\beta(x) = \begin{cases} \beta x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \beta - \beta x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$



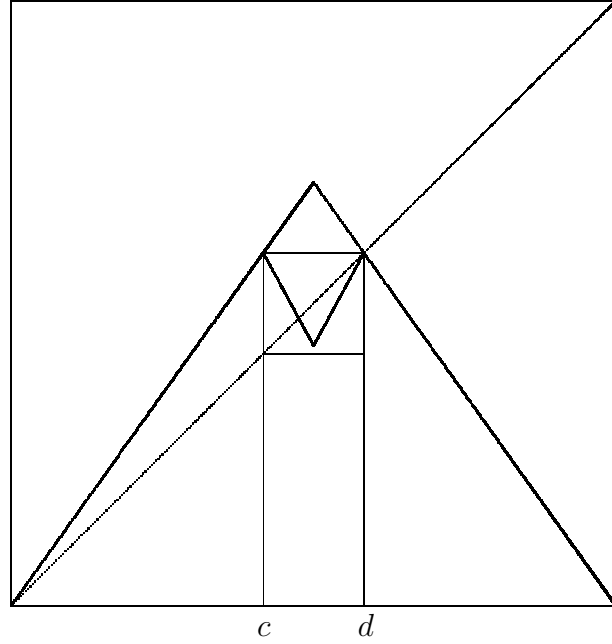
By Theorem 3 $h(u_\beta) = \log \beta$ for each $\beta \in (1, 2]$. In particular, if $\alpha, \beta \in (1, 2]$ with $\alpha \neq \beta$ then u_α and u_β are not conjugate. (If $f, g \in M(I)$ are conjugate then $\ell(f^n) = \ell(g^n)$ for all $n \geq 1$, and so $h(f) = h(g)$.)

Note if (ψ, g) is a reduction of $f \in S$ (with $g \in M([0, 1])$) then in fact $g \in S$. Thus if $f \in S$ with $h(f) > 0$ then by Theorem 2 there exists $\psi \in V([0, 1])$ such that $\psi \circ f = u_\beta \circ \psi$, where $\beta = \log h(f)$. Moreover, if f is essentially transitive then by Theorem 1 f and u_β are conjugate. In fact, the following result shows that the converse is true (since if f and g are conjugate and f is essentially transitive then so is g).

Proposition 3 *The mapping u_β is essentially transitive for each $\beta \in (1, 2]$.*

Proof Suppose first that $\beta \in (\sqrt{2}, 2]$. Let C be a u_β -cycle with period m , let B be one of the m components of C and let g be the restriction of f^m to B . Then g is uniformly piecewise linear with slope β^m and, since g has only one turning point, $\beta^m \leq 2$. This is only possible if $m = 1$, i.e. there are no u_β -cycles with period $m > 1$. Now let $J = [u_\beta^2(\frac{1}{2}), u_\beta(\frac{1}{2})]$; then $u_\beta(J) = J$, and so J is a u_β -cycle with period 1. Moreover, if K is any u_β -cycle with period 1 then $J \subset K$, since $\frac{1}{2} \in K$. Thus if K is any u_β -cycle with $K \subset J$ then $K = J$, and from this it is straightforward to show that J is transitive. (If J is not transitive then there exists a closed u_β -invariant subset F of J with $\text{int}(F) \neq \emptyset$ and $F \neq J$. Then $U = \text{int}(F)$ is a non-empty u_β -almost-invariant subset of J . An argument similar to that employed in the proof of Lemma 5 shows that J contains a periodic

component which can be used to define a u_β -cycle $K \subset F$, and this contradicts the fact that if K is any u_β -cycle with $K \subset J$ then $K = J$.) But it is clear that $[0, 1] \setminus E(J^o, u_\beta) \subset \{0, 1\}$, which shows that u_β is essentially transitive. Now suppose that $\beta \in (1, \sqrt{2}]$; then $d = (1 + \beta)^{-1}\beta$ is the unique fixed point of u_β in $(\frac{1}{2}, 1)$. Let $c = 1 - d$; thus $u_\beta(c) = d$. Then $u_\beta^2([c, d]) \subset [c, d]$, and it is easy to see that the restriction of u_β^2 to $[c, d]$ is conjugate to u_{β^2} .



But $u_\beta^m(x) \in [c, d]$ for some $m \geq 0$ for each $x \in (0, 1)$, and for each $x \in [0, 1]$ the set $\{y \in [0, 1] : u_\beta^n(y) = x \text{ for some } n \neq 0\}$ is countable; it thus follows that if u_{β^2} is essentially transitive then so is u_β . Therefore u_β is essentially transitive for each $\beta \in (1, 2]$. \square

If $\beta \in (0, 1]$ then u_β is certainly not essentially transitive: In this case it is easy to see that $Z_*(u_\beta) = [0, 1]$, and so there is no transitive u_β -cycle.

Let $\beta \in (1, 2]$; then u_β is essentially transitive and thus there exists a transitive u_β -cycle C such that $[0, 1] \setminus E(C^o, u_\beta)$ is countable. The proof of Proposition 3 in fact shows that C has period 2^p , where $p \geq 0$ is the smallest integer such that $2^{p+1} \log \beta > \log 2$. Thus C has period 1 if $\beta \in (\sqrt{2}, 2]$, period 2 if $\beta \in (\sqrt[4]{2}, \sqrt{2}]$, period 4 if $\beta \in (\sqrt[8]{2}, \sqrt[4]{2}]$ and so on. The same then holds true for an essentially transitive mapping $f \in S$.

We end this note by giving a proof of Theorem 2. The proof is essentially that to be found in Milnor and Thurston [5], [6] but without using any complex analysis. Fix a mapping $f \in M(I)$ with $h(f) > 0$ and put $r = \exp(-h(f))$; thus $r = 1/\beta$

and $0 < r < 1$. By Lemma 1 $\beta = \lim_{n \rightarrow \infty} \ell(f^n)^{1/n}$, and hence r is the radius of convergence of the power series $\sum_{n \geq 0} \ell(f^n) t^n$; in particular this means that the series $L(t) = \sum_{n \geq 0} \ell(f^n) t^n$ converges for all $t \in (0, r)$.

Lemma 6 $\lim_{t \uparrow r} L(t) = \infty$.

Proof By definition $\ell(f^n) \geq (\exp(h(f)))^n = \beta^n$ for each $n \geq 0$ and it therefore follows that $L(t) \geq \sum_{n \geq 0} (\beta t)^n = r(r - t)^{-1}$ for all $t \in (0, r)$. \square

Let \mathcal{J} denote the set of non-trivial closed intervals $J \subset I$; for $J \in \mathcal{J}$ and $n \geq 0$ denote by $\ell(f^n|J)$ the number of laps of f^n which intersect the interior of J (and so in fact $\ell(f^n|J) = \#(T(f^n) \cap \text{int}(J)) + 1$). Then $\ell(f^n|J) \leq \ell(f^n|I) = \ell(f^n)$, and thus in particular the series $L(J, t) = \sum_{n \geq 0} \ell(f^n|J) t^n$ converges for all $t \in (0, r)$. Now, noting that $L(I, t) = L(t) \neq 0$, put $\Lambda(J, t) = L(J, t)/L(I, t)$ for each $t \in (0, r)$, thus $0 \leq \Lambda(J, t) \leq 1$, because $L(J, t) \leq L(I, t)$.

Lemma 7 Let $J, K \in \mathcal{J}$ intersect in a single point. Then

$$\lim_{t \uparrow r} |\Lambda(J \cup K, t) - \Lambda(J, t) - \Lambda(K, t)| = 0.$$

Proof For each $n \geq 0$

$$\ell(f^n|J) + \ell(f^n|K) - 1 \leq \ell(f^n|J \cup K) \leq \ell(f^n|J) + \ell(f^n|K),$$

and thus

$$\begin{aligned} |\Lambda(J \cup K, t) - \Lambda(J, t) - \Lambda(K, t)| &= L(I, t)^{-1} |L(J \cup K, t) - L(J, t) - L(K, t)| \\ &\leq L(I, t)^{-1} \sum_{n \geq 0} t^n = (L(t)(1 - t))^{-1}. \end{aligned}$$

But by Lemma 6 $\lim_{t \uparrow r} (L(t)(1 - t))^{-1} = 0$ (since $r < 1$). \square

Lemma 8 Let $J \in \mathcal{J}$ be such that f is monotone on J . Then

$$\lim_{t \uparrow r} |r\Lambda(f(J), t) - \Lambda(J, t)| = 0.$$

Proof Since f is monotone on J it follows that $\ell(f^{n+1}|J) = \ell(f^n|f(J))$ for each $n \geq 0$, and thus

$$\begin{aligned} L(J, f) &= \sum_{n \geq 0} \ell(f^n|J) t^n = 1 + \sum_{n \geq 0} \ell(f^{n+1}|J) t^{n+1} \\ &= 1 + t \sum_{n \geq 0} \ell(f^n|f(J)) t^n = 1 + tL(f(J), t). \end{aligned}$$

Hence

$$\begin{aligned}
|r\Lambda(f(J), t) - \Lambda(J, t)| &= L(I, t)^{-1} |rL(f(J), t) - L(J, t)| \\
&\leq L(I, t)^{-1} (|rL(f(J), t) - tL(f(J), t)| + |tL(f(J), t) - L(J, t)|) \\
&\leq |r - t| + L(I, t)^{-1} = |r - t| + L(t)^{-1},
\end{aligned}$$

and by Lemma 6 $\lim_{t \uparrow r} (|r - t| + L(t)^{-1}) = 0$. \square

Lemma 9 *Let $J \in \mathcal{J}$. If f^m is monotone on J then $\limsup_{t \uparrow r} \Lambda(J, t) \leq r^m$.*

Proof Since f is monotone on $f^k(J)$ for each $k = 0, \dots, m-1$ it follows from Lemma 8 that

$$\lim_{t \uparrow r} |r^{k+1} \Lambda(f^{k+1}(J), t) - r^k \Lambda(f^k(J), t)| = \lim_{t \uparrow r} r^k |r \Lambda(f(f^k(J)), t) - \Lambda(f^k(J), t)| = 0.$$

Hence $\lim_{t \uparrow r} |r^m \Lambda(f^m(J), t) - \Lambda(J, t)| = 0$, and so

$$\limsup_{t \uparrow r} \Lambda(J, t) = \limsup_{t \uparrow r} r^m \Lambda(f^m(J), t) \leq r^m,$$

since $\Lambda(f^m(J), t) \leq 1$ for all $t \in (0, r)$. \square

Lemma 10 *There exists a sequence $\{t_n\}_{n \geq 1}$ from $(0, r)$ with $\lim_{n \rightarrow \infty} t_n = r$ such that $\{\Lambda(J, t_n)\}_{n \geq 1}$ converges for all $J \in \mathcal{J}$.*

Proof Let I_o be a countable dense subset of I with $\{a, b\} \subset I_o$ and $T(f^n) \subset I_o$ for each $n \geq 1$; let \mathcal{J}_o be the set of intervals $J = [c, d] \in \mathcal{J}$ such that $c, d \in I_o$, thus \mathcal{J}_o is countable. Now if $J \in \mathcal{J}$ and $\{s_n\}_{n \geq 1}$ is any sequence from $(0, r)$ then the sequence $\{\Lambda(J, s_n)\}_{n \geq 1}$ is bounded, and so there exists a subsequence $\{n_k\}_{k \geq 1}$ such that $\{\Lambda(J, s_{n_k})\}_{k \geq 1}$ converges. Therefore, since \mathcal{J}_o is countable, a sequence $\{t_n\}_{n \geq 1}$ from $(0, r)$ with $\lim_{n \rightarrow \infty} t_n = r$ can be found (using the standard diagonal argument) such that $\{\Lambda(J, t_n)\}_{n \geq 1}$ converges for every $J \in \mathcal{J}_o$. In fact this sequence then converges for all $J \in \mathcal{J}$: First consider $J = [c, d] \in \mathcal{J}$ with $c \in I_o$ and $d \notin I_o$. Let $\varepsilon > 0$ and choose $m \geq 1$ so that $r^m < \varepsilon$; since $d \notin T(f^m) \cup \{a, b\}$ there exist $u, v \in I_o$ such that $c < u < d < v$ and f^m is monotone on $[u, v]$. Then $[c, u]$ and $[c, v]$ are both in \mathcal{J}_o , and $\Lambda([c, u], t) \leq \Lambda(J, t) \leq \Lambda([c, v], t)$ for all $t \in (0, r)$; hence by Lemmas 7 and 9

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \Lambda(J, t_n) - \liminf_{n \rightarrow \infty} \Lambda(J, t_n) &\leq \lim_{n \rightarrow \infty} \Lambda([c, v], t_n) - \lim_{n \rightarrow \infty} \Lambda([c, u], t_n) \\
&= \lim_{n \rightarrow \infty} \Lambda([u, v], t_n) \leq r^m < \varepsilon.
\end{aligned}$$

Therefore, since $\varepsilon > 0$ was arbitrary, the sequence $\{\Lambda(J, t_n)\}_{n \geq 1}$ converges. The same argument also gives that this sequence converges when $J = [c, d]$ with $c \notin I_o$ and $d \in I_o$. Finally, if $J = [c, d]$ and $c \notin I_o$, $d \notin I_o$ then choose $u \in I_o$ with $c < u < d$; then $\{\Lambda([c, u], t_n)\}_{n \geq 1}$ and $\{\Lambda([u, d], t_n)\}_{n \geq 1}$ both converge, and so by Lemma 7 $\{\Lambda(J, t_n)\}_{n \geq 1}$ converges. \square

Now fix a sequence $\{t_n\}_{n \geq 1}$ as in Lemma 10, and put $\Lambda(J) = \lim_{n \rightarrow \infty} \Lambda(J, t_n)$ for each $J \in \mathcal{J}$. Then by Lemmas 7, 8 and 9:

- (1) If $J, K \in \mathcal{J}$ intersect in a single point then $\Lambda(J \cup K) = \Lambda(J) + \Lambda(K)$.
- (2) If $J \in \mathcal{J}$ and f is monotone on J then $r\Lambda(f(J)) = \Lambda(J)$.
- (3) If $J \in \mathcal{J}$, $m \geq 1$ and f^m is monotone on J then $\Lambda(J) \leq r^m$.

Also define a mapping $\pi : I \rightarrow [0, 1]$ by letting $\pi(a) = 0$ and $\pi(x) = \Lambda([a, x])$ for all $x \in (a, b]$.

Lemma 11 *The mapping $\pi : I \rightarrow [0, 1]$ is continuous, increasing and surjective.*

Proof It is clear that π is increasing, and if it is continuous then it is surjective, because $\pi(a) = 0$ and $\pi(b) = \Lambda(I) = 1$. Let $x \in I$ and $\varepsilon > 0$; choose $m \geq 1$ so that $r^m < \varepsilon$. Then there exists $\delta > 0$ such that $\{w \in T(f^m) : |\delta - x| < \delta\} \subset \{x\}$. Now if $U = I \cap (x - \delta, x + \delta)$ then U is a neighbourhood of x in I , and it follows from (1) and (3) that $|\pi(y) - \pi(x)| < \varepsilon$ for all $y \in U$, since if $y > x$ (resp. $y < x$) then f^m is monotone on $[x, y]$ (resp. on $[y, x]$). This shows π is continuous. \square

Lemma 12 *There exists a unique mapping $\alpha : [0, 1] \rightarrow [0, 1]$ with $\pi \circ f = \alpha \circ \pi$.*

Proof If $\alpha : [0, 1] \rightarrow [0, 1]$ is a mapping with $\pi \circ f = \alpha \circ \pi$ then $\alpha(z) = \pi(f(x))$ whenever $x \in I$ is such that $\pi(x) = z$. Conversely, this relation can be used to define a mapping α with $\pi \circ f = \alpha \circ \pi$, provided $\pi(f(x)) = \pi(f(y))$ whenever $x, y \in I$ are such that $\pi(x) = \pi(y)$. Let $x, y \in I$ with $x < y$ and $\pi(x) = \pi(y)$, and consider u, v with $x \leq u < v \leq y$ so that f is monotone on $[u, v]$. Then $\pi(u) = \pi(v)$ and hence by (1) and (2)

$$r\Lambda(f([u, v])) = \Lambda([u, v]) = \Lambda([a, v]) - \Lambda([a, u]) = \pi(v) - \pi(u) = 0,$$

i.e., $\Lambda(f([u, v])) = 0$; thus $\pi(f(u)) = \pi(f(v))$, because $f(u)$ and $f(v)$ are the end-points of the interval $f([u, v])$. But $[u, v]$ can be written as

$$[x, y] = [u_0, u_n] = [u_0, u_1] \cup \cdots \cup [u_{n-1}, u_n]$$

with f monotone on each of the intervals $[u_j, u_{j+1}]$, $0 \leq j \leq n-1$, and therefore

$$\pi(f(x)) = \pi(f(u_0)) = \pi(f(u_1)) = \cdots = \pi(f(u_n)) = \pi(f(y)).$$

This shows that α exists. The uniqueness of α follows immediately from the fact that π is surjective. \square

Let $\alpha : [0, 1] \rightarrow [0, 1]$ be the unique mapping with $\pi \circ f = \alpha \circ \pi$.

Lemma 13 *The mapping α is uniformly piecewise linear with slope β .*

Proof Let $[c, d]$ be a lap of f on which f is increasing, and let $z \in (\pi(c), \pi(d))$. Then there exists $x \in (c, d]$ with $\pi(x) = z$, and hence by (1) and (2)

$$\begin{aligned} \alpha(z) - \alpha(\pi(c)) &= \alpha(\pi(x)) - \alpha(\pi(c)) = \pi(f(x)) - \pi(f(c)) \\ &= \Lambda([a, f(x)]) - \Lambda([a, f(c)]) = \Lambda([f(c), f(x)]) = \Lambda(f([c, x])) \\ &= \beta\Lambda([c, x]) = \beta(\Lambda([a, x]) - \Lambda([a, c])) \\ &= \beta(\pi(x) - \pi(c)) = \beta(z - \pi(c)) . \end{aligned}$$

This shows that α is linear with slope β on $[\pi(c), \pi(d)]$. If $[u, v]$ is a lap of f on which f is decreasing then a similar calculation shows that α is linear on $[\pi(u), \pi(v)]$ with slope $-\beta$. Therefore α is uniformly piecewise linear with slope β . \square

Now let $\gamma : [0, 1] \rightarrow I$ be the linear rescaling given by $\gamma(t) = a + (b - a)t$, and define $\psi, g : I \rightarrow I$ by $\psi = \gamma \circ \pi$ and $g = \gamma \circ \alpha \circ \gamma^{-1}$. Then $\psi \in V(I)$ and $g \in M(I)$ is uniformly piecewise linear with slope β (because α is); moreover,

$$\psi \circ f = \gamma \circ \pi \circ f = \gamma \circ \alpha \circ \pi = \gamma \circ \alpha \circ \gamma^{-1} \circ \gamma \circ \pi = g \circ \psi ,$$

i.e., (ψ, g) is a reduction of f . This completes the proof of Theorem 2. \square

Remark: In Milnor and Thurston [5], [6] it is shown that for each $J \in \mathcal{J}$ there exists a meromorphic function $L_1(J, \cdot) : D = \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{C} \cup \{\infty\}$ with $L_1(J, t) = L(J, t)$ for all $t \in (0, r)$. There thus also exists a meromorphic function $\Lambda_1(J, \cdot) : D \rightarrow \mathbb{C} \cup \{\infty\}$ with $\Lambda_1(J, t) = \Lambda(J, t)$ for all $t \in (0, r)$ (with of course $\Lambda_1(J, \cdot) = L_1(J, \cdot)/L_1(I, \cdot)$). Now since $0 \leq \Lambda(J, t) \leq 1$ for all $t \in (0, r)$, it follows that $0 \leq \Lambda_1(J, r) \leq 1$ and $\lim_{t \uparrow r} \Lambda(J, t) = \Lambda_1(J, r)$. In particular, $\Lambda(J) = \Lambda_1(J, r)$. The construction in Lemma 10 is therefore not really necessary.

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